

NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
POLITEHNICA BUCHAREST
FACULTY OF APPLIED SCIENCES

MARILENA CIONTESCU

Ph.D. Thesis Summary:

**Contributions to Multiobjective and
Interval-Valued Optimization Models**

Scientific Supervisor:

Prof. univ. dr. Savin Treanță

Bucharest, 2026

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Abstract

The present thematic proposal focuses on control theory and the investigation of multiobjective and interval-valued optimization problems. Our aim is to develop the research study in this area by formulating and demonstrating some results that characterize well-defined and efficient solutions for new classes of variational problems governed by multiple integrals and mixed constraints, under both convexity and non-convexity assumptions. In addition, duality as one of the central pillars of modern optimization, plays a key role in our analysis. We establish various relationships between dual models (Wolfe and Mond-Weir) and the original variational control problem, providing a more complete and efficient framework for analyzing and solving these optimization problems. Others approaches developed in this Thesis are the exact l_1 penalty function method and the modified cost function technique.

In the first part of this Thesis, we focus on the interval-valued variational optimization problem governed by control variables. We explore the relationship between (strong) LU-optimal solutions associated with our variational control problem and the solutions of some (weak, split) controlled variational inequalities, under assumptions of convexity for the involved integral functionals. We extend our research on the interval-valued optimal control problems in the context of invex integral functionals. Necessary and sufficient conditions of optimality are stated for the considered optimization problem. Moreover, various duality relationships between the dual models (Wolfe and Mond-Weir) and the primal optimization problem are deliberated.

The field of multi-dimensional controlled optimization problem is an active field in operations research, especially the multi-dimensional optimization problem with controlled dynamical system constraints. Therefore, our attention is turned to this topic. We construct an equivalent unconstrained optimization model, using the exact l_1 penalty function method associated with our problem, then establish that both the constrained optimization model and its associated unconstrained optimization model attain their efficiency at the same point under the convexity assumption.

In the second part of this Thesis, in the study of multi-dimensional optimization problems, non-convex controlled minimization models play an important role and represent a developing field of operations research. Motivated by previous studies, in this Thesis we study and characterize the solution set of a multi-cost extremization problem, governed by invex, pseudoinvex, or quasiinvex multiple integral type functionals. The main contribution of this chapter is the multi-dimensional framework on which the problem is built. We also prove a duality connection between our multi-cost extremization problem and its associated mixed multi-cost dual model.

Building on current progress in multi-cost optimization, we extend vector variational inequalities, in Minty and Stampacchia forms, to the controlled variational inequalities of vector type. More precisely, we incorporate a new variable to these classes of variational problems, namely, the control function, and establish various connections between the considered classes of variational problems.

The line of research continues with a new investigation related to multiobjective programming. We examine a family of non-convex and non-linear multiobjective variational control models with the technique of modified cost functional, under univexity hypotheses. Concretely, we state some connections between (weakly) efficient points of these multiobjective variational control models.

Introduction

Control Theory is one of the topics addressed in this Thesis, being a developing field in operations research, and plays a crucial role in investigating uncertainty in variational control problems. Another topic investigated in this Thesis represents *Optimization Theory*, which plays an important role in Control Theory. Control Theory and Optimization Theory are closely related because both try to make a system as best as possible. In fact, many control problems can be viewed as optimization problems. When limits and restrictions are present, control naturally becomes an optimization problem in which we must choose the best possible decision satisfying the imposed constraints. Optimization is the process of determining the best advantageous solutions to a problem by maximizing or minimizing a well-defined objective under a given set of constraints. Optimization exists everywhere in this world. The concept of optimization is not new. From one of the earliest recorded examples of optimization, a legendary problem involving Queen Dido in 814 BC, who maximized the land area using a bull's hide, to advanced artificial intelligence (AI) systems, there is a long history. Many philosophers and mathematicians have contributed to this field, even in ancient times, starting with Ancient Greece and Rome, aiming to solve a wide variety of real-life problems and continuing nowadays as an active area research in theoretical and computational methods. Since ancient times, Euclid and Archimedes have approached optimization in geometry problems, but without using modern formulas. We could say that the beginning of mathematical optimization started in the 17th century, when Isaac Newton (1643-1727) and Gottfried Wilhelm Leibniz (1646-1716) studied problems of maximum and minimum analytically. Later, Joseph-Louis Lagrange (1736-1813) systematized and generalized methods for constrained optimization. In mid-18th century, Euler laid the foundations of the calculus of variations by deriving the Euler–Lagrange condition for optimality of functionals. However, the start of true mathematical rigor in optimization started in the 19th century, when Augustin Louis Cauchy (1789-1857) and Karl Weierstrass (1815-1897) established the conditions for the existence of the optimum. At the end of the 19th century, A.M. Lyapunov (1857-1918) studied the stability of a system and formulated the mathematical conditions for stable evolution. A major development occurred in the 20th century, when Leonid Kantorovich (1912-1986), in 1939, introduced linear programming as a technique for solving optimization problems defined by linear equations, and later, George Dantzig, in 1947, introduced the simplex method, representing a major impact on the solution of linear programming problems due to massive applications in economics, engineering and computer science. George Dantzig (1914-2005) is known for his work as the father of linear programming and the “simplex algorithm”. He stated that “True optimization is the revolutionary contribution of modern research to decision processes.”. The concept of linear control appeared in the 1930–1940s, and later, in 1956, Pontryagin introduced the Maximal Principle for Dynamical Systems, which represents the first generalized mathematical formulation on optimization of evolution. In 1957, Bellman generalized the Euler–Lagrange method to optimal control with constraints and variable final state. The direct link between stability and optimization became clear in the 1960s with the emergence of Linear Optimal Control, and was further developed in the 1970s and 1980s through Non-linear and Robust Control. From 1990 to the present, we can talk about Constrained Optimal Control, Predictive Control and AI. In this Thesis, we explore, among others, an advanced topic, *Duality Theory*. Duality theory evolves from Lagrange multipliers to robust control and AI, keeping the same idea: primal problem, evolution of a system and dual problem, conditions for optimizing this evolution. In duality theory, two problems are associated in such a way that the existence

of a solution to one problem ensures the solution of the other one under certain assumptions. Therefore, duality is basically a unifying principle that develops the relationship between two constrained optimization problems. Control theory has evolved from simple mechanisms to abstract mathematical formulations and, further, to intelligent autonomous systems. Nowadays, we talk about optimization in complex systems, machine learning and AI. Evolution is very fast, and current systems are much more complex. Therefore, the classical theory must be constantly improved. So, we can say that with the increase in complexity of modern systems, uncertainties appear and a lot of performance and efficiency are needed. In other words, the problems grow in complexity and the solutions must be fast and precise to solve these problems. As a result, there arises the necessity for innovative and more efficient optimization methods.

Research Aim and Specific Objectives

The *main objective* of this Thesis is to advance research in control theory and optimization problems governed by simple and multiple integral functionals under different constraints, through innovative optimization techniques and approaches, and to improve duality theory by discovering new relationships between dual models (Wolfe and Mond-Weir) and the original variational control problems. The main purpose is to improve the performance and efficiency of the analyzed systems and also to offer a new start for future research.

As *specific objectives* we mention:

- 1) to investigate the relationships between a class of interval-valued optimal control problems and the associated inequalities;
- 2) to state some results on solutions associated with optimal control problems with generalized invariant convex interval-valued functionals;
- 3) to characterize the solution set of a non-convex multi-cost extremization problem;
- 4) to explore new relationships between the dual models and the initial optimization problem;
- 5) to investigate an interval-valued multi-dimensional controlled optimization model for vector case and to provide techniques to find the solution by transforming it into an equivalent unconstrained model;
- 6) to extend vector variational inequalities, in Minty and Stampacchia forms, and to establish a range of connections between the given multiple cost optimization models adjusting the concepts of (strictly) strong convexity and preconvexity;
- 7) to apply the technique of modified cost functional to a multiobjective variational control model determined by univex functionals and to analyze the relationships between the original and modified models.

Thesis Summary

The present Thesis is organized in 6 Chapters. Each chapter provides background information, research motivation, and concluding remarks.

The principal purpose of **Chapter 1**, titled *Connections between interval-valued variational control problems and the associated inequalities*, is to analyze some connections between interval-valued variational control problems and the associated inequalities, under convexity assumptions.

We define the interval-valued functionals generated by multiple integrals. Also, we introduce (strictly) convexity for real-valued multiple integral type functionals. In addition, we state (strictly) LU-convexity of interval-valued multiple integral type functionals.

We formulate an interval-valued variational problem as follows:

$$\begin{aligned}
 (P) \quad & \min_{(y(\cdot), z(\cdot))} \int_M g(t, y(t), y_\gamma(t), z(t)) d\zeta \\
 & \text{subject to} \\
 & K_\delta(t, y(t), y_\gamma(t), z(t)) \leq 0, \quad \delta = \overline{1, l} \\
 & T_\eta(t, y(t), y_\gamma(t), z(t)) := \frac{\partial y}{\partial t^\gamma}(t) - Q_\eta(t, y(t), z(t)) = 0, \quad \eta = \overline{1, r} \\
 & y|_{\partial M} = y_0 = \text{given},
 \end{aligned}$$

where $t \in M$, $g : M \times Y \times Y \times Z \mapsto \mathcal{F}$, $K_\delta : M \times Y \times Y \times Z \mapsto \mathbb{R}$ and $T_\eta : M \times Y \times Y \times Z \mapsto \mathbb{R}$ are C^1 -class functions.

We denote by

$$\begin{aligned}
 S = \{ & (y, z) \in Y \times Z \mid K_\delta(t, y(t), y_\gamma(t), z(t)) \leq 0, \\
 & T_\eta(t, y(t), y_\gamma(t), z(t)) = 0, \ y|_{\partial M} = y_0 = \text{given} \}
 \end{aligned}$$

the feasible solution set to (P) and consider that S is a convex subset of $Y \times Z$.

We define an LU-optimal solution for this problem.

Definition 1. A point $(y^0, z^0) \in S$ is said to be a (strong) LU-optimal solution to (P) if

$$\int_M g(\pi^0) d\zeta \preceq_{LU} (\prec_{LU}) \int_M g(\pi) d\zeta$$

for all $(y, z) \in S$.

Now, in order to obtain some characterization results for the interval-valued optimization problem (P), we state the following inequalities:

(i) find $(y^0, z^0) \in S$ such that there exists no $(y, z) \in S$, satisfying

$$\begin{aligned}
 (I) \quad & \int_M \{g_y^L(\pi^0) + g_y^U(\pi^0)\} (y - y^0) d\zeta \\
 & + \int_M \{g_z^L(\pi^0) + g_z^U(\pi^0)\} (z - z^0) d\zeta \\
 & + \int_M \{g_{y_\gamma}^L(\pi^0) + g_{y_\gamma}^U(\pi^0)\} D_\gamma(y - y^0) d\zeta \leq 0;
 \end{aligned}$$

(ii) find $(y^0, z^0) \in S$ such that there exists no $(y, z) \in S$, satisfying

$$(WI) \quad \int_M \{g_y^L(\pi^0) + g_y^U(\pi^0)\} (y - y^0) d\zeta \\ + \int_M \{g_z^L(\pi^0) + g_z^U(\pi^0)\} (z - z^0) d\zeta \\ + \int_M \{g_{y_\gamma}^L(\pi^0) + g_{y_\gamma}^U(\pi^0)\} D_\gamma (y - y^0) d\zeta < 0;$$

(iii) find $(y^0, z^0) \in S$ such that, for all $(y, z) \in S$, the inequalities

$$(SI) \quad \int_M g_y^L(\pi^0) (y - y^0) d\zeta + \int_M g_z^L(\pi^0) (z - z^0) d\zeta \\ + \int_M g_{y_\gamma}^L(\pi^0) D_\gamma (y - y^0) d\zeta > 0, \\ \int_M g_y^U(\pi^0) (y - y^0) d\zeta + \int_M g_z^U(\pi^0) (z - z^0) d\zeta \\ + \int_M g_{y_\gamma}^U(\pi^0) D_\gamma (y - y^0) d\zeta > 0$$

are satisfied.

In the following, by considering the solutions of the considered (weak, split) inequalities (I), (WI) and (SI), we establish some existence results for the (strong) LU -optimal solutions of the associated interval-valued variational problem (P).

Next, by using the solutions associated with the inequality (I), we provide a sufficient condition such that $(y^0, z^0) \in S$ is an LU -optimal solution of (P).

Theorem 1. *Let $(y^0, z^0) \in S$ be a solution to (I) and functional $\int_M g(\pi) d\zeta$ is LU -convex at $(y^0, z^0) \in S$. Then $(y^0, z^0) \in S$ is an LU -optimal solution to (P).*

The reciprocal of the previous theorem is given below.

Theorem 2. *Let $(y^0, z^0) \in S$ be an LU -optimal solution of (P) and $\int_M -g(\pi) d\zeta$ is strictly LU -convex at $(y^0, z^0) \in S$. Then $(y^0, z^0) \in S$ is a solution of (I).*

By considering the solutions of the inequality (SI), we state a sufficient condition such that a pair $(y^0, z^0) \in S$ is a strong LU -optimal solution of (P).

Theorem 3. *Let $(y^0, z^0) \in S$ fulfils (SI) and $\int_M g(\pi) d\zeta$ is LU -convex at $(y^0, z^0) \in S$. Then $(y^0, z^0) \in S$ is a strong LU -optimal solution for (P).*

Next, we provide the conditions such that the reciprocal of the previous theorem is fulfilled.

Theorem 4. Let $(y^0, z^0) \in S$ be a strong LU -optimal solution of (P) and $\int_M -g(\pi)d\zeta$ is LU -convex at $(y^0, z^0) \in S$. Then $(y^0, z^0) \in S$ satisfies (SI) .

The following theorem shows us that LU -convexity at $(y^0, z^0) \in S$ of $\int_M -g(\pi)d\zeta$, together with $(y^0, z^0) \in S$ is an LU -optimal solution of (P) , make sure that $(y^0, z^0) \in S$ solves the weak variational control inequality (WI) .

Theorem 5. Consider $(y^0, z^0) \in S$ is an LU -optimal solution of (P) and $\int_M -g(\pi)d\zeta$ is LU -convex at $(y^0, z^0) \in S$. Then $(y^0, z^0) \in S$ solves the weak variational control inequality (WI) .

The next result is the converse result of the previous theorem.

Theorem 6. Let $(y^0, z^0) \in S$ solves the weak variational control inequality (WI) and $\int_M g(\pi)d\zeta$ is strictly LU -convex at $(y^0, z^0) \in S$. Then $(y^0, z^0) \in S$ is an LU -optimal solution of (P) .

We provide an illustrative application associated with the above-mentioned derived theoretical results.

The methodology used is based on several techniques from the calculus of variations and control theory, which are appropriate in the study of the considered interval-valued variational control problems. To the best of the authors' knowledge, the derived theoretical results for such types of problems are new in the specialized literature.

Regarding the future research directions associated with this study, we could mention investigation of the case where the second-order partial derivatives are presented, as well as the situation when the involved functionals are not necessarily (strictly) convex.

The content of this chapter is based on the paper *On some connections between interval-valued variational control problems and the associated inequalities*, authored by Ciontescu, M., Treanță, S., published in *Results in Control and Optimization*, Vol. 12, 100300, 2023; DOI: 10.1016/j.rico.2023.100300; WOS: 001209054000005; eISSN: 2666-7207 [30].

In **Chapter 2**, titled *Optimal control problems governed by invariant convex interval-valued functionals*, we extend our research on interval-valued optimal control problems, but under the assumptions of invex functionals.

We present and analyze some results on solutions associated with interval-valued optimal control problems, determined by generalized invariant convex functionals. We define invexity, pseudoinvexity, and quasi-invexity.

We investigate the primal optimization problem driven by an interval-valued cost functional:

$$(P) \quad \min_{(x,u) \in \mathcal{X} \times \mathcal{U}} \left\{ K(x, u) = [K^L(x, u), K^U(x, u)] \right\}$$

subject to

$$g_j(t, x, u) \leq 0, \quad t \in [a, b], \quad j = 1, 2, \dots, m$$

$$\frac{dx_i}{dt} = h_i(t, x, u), \quad t \in [a, b], \quad i = 1, 2, \dots, n$$

$$x(a) = \alpha, \quad x(b) = \beta,$$

where $K : \mathcal{X} \times \mathcal{U} \rightarrow I$ is a C^1 -class interval-valued functional,

$$K^L(x, u) := \int_a^b f^L(t, x(t), u(t))dt, \quad K^U(x, u) := \int_a^b f^U(t, x(t), u(t))dt,$$

with $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow I$, $f = [f^L, f^U]$, a differentiable interval-valued function, $g_j : [a, b] \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$, $j = 1, 2, \dots, m$, a differentiable real-valued function, and $\alpha, \beta \in \mathbb{R}$ as fixed real numbers.

We introduce the concept of optimal solution for the considered variational control problem. We denote by Ω the set of all feasible solutions of (P).

Definition 2. The pair $(x^*, u^*) \in \Omega$ is an *LU-optimal solution* of problem (P) if there exists no $(x_0, u_0) \in \Omega$ satisfying $K(x_0, u_0) <_{LU} K(x^*, u^*)$.

We present the Karush-Kuhn-Tucker type necessary optimality conditions associated with our problem (P).

Theorem 7. *If $(x^*, u^*) \in \Omega$ is an LU-optimal solution of primal variational control problem (P) and the constraint functions satisfy constraint qualification at (x^*, u^*) , then there exist multipliers (functions on $[a, b]$) $0 < \gamma^L, \gamma^U \in \mathbb{R}$, $0 \leq \zeta_j \in \mathbb{R}$, $j = 1, 2, \dots, m$, and $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, such that the next relations*

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial x_i}(t, x^*, u^*) + \gamma^U \frac{\partial f^U}{\partial x_i}(t, x^*, u^*) + \sum_{j=1}^m \zeta_j \frac{\partial g_j}{\partial x_i}(t, x^*, u^*) \\ & + \sum_{i=1}^n \lambda_i \frac{h_i}{\partial x_i}(t, x^*, u^*) + \frac{d\lambda_i}{dt}(t) = 0, \quad i = \overline{1, n} \end{aligned} \quad (1)$$

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial u_s}(t, x^*, u^*) + \gamma^U \frac{\partial f^U}{\partial u_s}(t, x^*, u^*) + \sum_{j=1}^m \zeta_j \frac{\partial g_j}{\partial u_s}(t, x^*, u^*) \\ & + \sum_{i=1}^n \lambda_i \frac{h_i}{\partial u_s}(t, x^*, u^*) = 0, \quad s = \overline{1, p} \end{aligned} \quad (2)$$

$$\zeta_j g_j(t, x^*, u^*) = 0, \quad j = \overline{1, m} \quad (3)$$

are fulfilled for all $t \in [a, b]$, except at discontinuities.

We formulate and prove two results on sufficient conditions of optimality for (P).

By considering the invexity, pseudoinvexity, and quasi-invexity assumptions of the involved functionals, the following results establish sufficient conditions for optimality in problem (P).

Theorem 8. *Let $(x^*, u^*) \in \Omega$, the constraint functions satisfy the suitable Kuhn-Tucker constraint qualification at (x^*, u^*) , and the relations (1)-(3) from Theorem 7 are satisfied for $t \in [a, b]$, except at discontinuities. Moreover, we assume that K^L and K^U are $(q, w) - \pi_1$ -invex and $(q, w) - \pi_2$ -invex at (x^*, u^*) with respect to τ, ν and σ , respectively, and $G(x, u) := \int_a^b \sum_{j=1}^m \zeta_j g_j(t, x, u)dt$ and $H(x, u) := \int_a^b \sum_{i=1}^n \lambda_i [h_i(t, x, u) - \dot{x}_i]dt$ are $(q, w) - \pi_3$ -invex and $(q, w) - \pi_4$ -invex at (x^*, u^*) with respect to τ, ν and σ , respectively. If $\gamma^L \pi_1 + \gamma^U \pi_2 + \pi_3 + \pi_4 \geq 0$, then (x^*, u^*) is an LU-optimal solution to the problem (P).*

Theorem 9. Let $(x^*, u^*) \in \Omega$, the constraint functions satisfy the suitable Kuhn-Tucker constraint qualification at (x^*, u^*) , and the relations (1)-(3) from Theorem 7 are satisfied for $t \in [a, b]$, except at discontinuities. Moreover, we consider $\gamma^L K^L + \gamma^U K^U$ is $(q, w) - \pi_1$ -pseudoinvex at (x^*, u^*) with respect to τ, ν and σ , and $G(x, u) := \int_a^b \sum_{j=1}^m \zeta_j g_j(t, x, u) dt$ and $H(x, u) := \int_a^b \sum_{i=1}^n \lambda_i [h_i(t, x, u) - \dot{x}_i] dt$ are $(q, w) - \pi_2$ -quasiinvex and $(q, w) - \pi_3$ -quasiinvex at (x^*, u^*) with respect to τ, ν and σ , respectively. If $\pi_1 + \pi_2 + \pi_3 \geq 0$, then (x^*, u^*) is an LU-optimal solution to the problem (P).

In the following we consider the following Wolfe type dual problem associated with the primal optimization problem (P):

$$(D - W) \quad \max_{(z, r) \in \mathcal{X} \times \mathcal{U}} \left\{ K(z, r) + G(z, r) + H(z, r) \right\}$$

subject to

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial x_i}(t, z, r) + \gamma^U \frac{\partial f^U}{\partial x_i}(t, z, r) + \sum_{j=1}^m \zeta_j \frac{\partial g_j}{\partial x_i}(t, z, r) \\ & + \sum_{i=1}^n \lambda_i \frac{h_i}{\partial x_i}(t, z, r) + \frac{d\lambda_i}{dt}(t) = 0, \quad i = \overline{1, n} \end{aligned} \quad (4)$$

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial u_s}(t, z, r) + \gamma^U \frac{\partial f^U}{\partial u_s}(t, z, r) + \sum_{j=1}^m \zeta_j \frac{\partial g_j}{\partial u_s}(t, z, r) \\ & + \sum_{i=1}^n \lambda_i \frac{h_i}{\partial u_s}(t, z, r) = 0, \quad s = \overline{1, p} \end{aligned} \quad (5)$$

$$z(a) = \alpha, \quad z(b) = \beta, \quad \gamma^L + \gamma^U = 1, \quad (6)$$

We present some Wolfe type duality results (weak duality, strong duality, strict converse duality) as given below:

Theorem 10. (Weak duality) Suppose that (x, u) and $(z, r, \gamma^L, \gamma^U, \zeta, \lambda)$ are some feasible points to (P) and (D-W), respectively. Also, consider that $\gamma^L > 0, \gamma^U > 0$ and $\zeta_j \geq 0, j = \overline{1, m}, \lambda_i \in \mathbb{R}, i = \overline{1, n}$, such that the functional $\gamma^L K^L + \gamma^U K^U + G + H$ is $(q, w) - \pi$ -invex at (z, r) with respect to τ, ν and σ , with $\pi \geq 0$. Then

$$K(x, u) \geq_{LU} K(z, r) + G(z, r) + H(z, r).$$

Theorem 11. (Strong duality) If (x^*, u^*) is an LU-optimal solution for (P), then there exist the real scalars $\gamma^{*L} > 0, \gamma^{*U} > 0, \zeta^* \geq 0$ and $\lambda^* \in \mathbb{R}$ such that $(x^*, u^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ becomes a feasible solution for (D-W). In addition, the two objective functionals have the same value. Moreover, if Theorem 10 is satisfied for all feasible points $(z^*, r^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$, then $(x^*, u^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ is an LU-optimal solution of (D-W).

Theorem 12. (Strict converse duality) *Let (x^*, u^*) , $(z^*, r^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ are the feasible solutions to (P) and (D-W), respectively. Assume that $\gamma^{*L}K^L + \gamma^{*U}K^U + G + H$ is strictly $(q, w) - \pi$ -invex at (z^*, r^*) with respect to τ, ν and σ , with $\pi \geq 0$, and*

$$\begin{aligned} & \gamma^{*L}K^L(x^*, u^*) + \gamma^{*U}K^U(x^*, u^*) + \int_a^b \sum_{j=1}^m \zeta_j^* g_j(t, x^*, u^*) dt \\ & + \int_a^b \sum_{i=1}^n \lambda_i^* [h_i(t, x^*, u^*) - \dot{x}_i^*] dt \\ & \leq \gamma^{*L}K^L(z^*, r^*) + \gamma^{*U}K^U(z^*, r^*) + \int_a^b \sum_{j=1}^m \zeta_j^* g_j(t, z^*, r^*) dt \\ & + \int_a^b \sum_{i=1}^n \lambda_i^* [h_i(t, z^*, r^*) - \dot{z}_i^*] dt. \end{aligned} \quad (7)$$

Then $(x^*, u^*) = (z^*, r^*)$.

Next, we formulate the Mond-Weir type dual problem associated with the primal optimization problem (P):

$$(D-MW) \quad \max_{(z, r) \in \mathcal{X} \times \mathcal{U}} \left\{ K(z, r) = [K^L(z, r), K^U(z, r)] \right\}$$

subject to

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial x_i}(t, z, r) + \gamma^U \frac{\partial f^U}{\partial x_i}(t, z, r) + \sum_{j=1}^m \zeta_j \frac{\partial g_j}{\partial x_i}(t, z, r) \\ & + \sum_{i=1}^n \lambda_i \frac{h_i}{\partial x_i}(t, z, r) + \frac{d\lambda_i}{dt}(t) = 0, \quad i = \overline{1, n} \end{aligned} \quad (8)$$

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial u_s}(t, z, r) + \gamma^U \frac{\partial f^U}{\partial u_s}(t, z, r) + \sum_{j=1}^m \zeta_j \frac{\partial g_j}{\partial u_s}(t, z, r) \\ & + \sum_{i=1}^n \lambda_i \frac{h_i}{\partial u_s}(t, z, r) = 0, \quad s = \overline{1, p} \end{aligned} \quad (9)$$

$$z(a) = \alpha, \quad z(b) = \beta, \quad \gamma^L + \gamma^U = 1 \quad (10)$$

$$\zeta_j g_j(t, z, r) \geq 0, \quad j = \overline{1, m}. \quad (11)$$

We define an LU-optimal solution of (D-MW).

Definition 3. Let $(z^*, r^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ be a feasible point of (D-MW). We say that $(z^*, r^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ is an *LU-optimal solution of (D-MW)* if there exists no $(z, r, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ such that $K(z^*, r^*) <_{LU} K(z, r)$.

Moreover, weak, strict converse and strong duality relationships between the dual model (D-MW) and the primal optimization problem (P) are deliberated.

Theorem 13. (Weak duality) Suppose that (x, u) and $(z, r, \gamma^L, \gamma^U, \zeta, \lambda)$ are some feasible points to (P) and $(D-MW)$, respectively. Further, consider that $\gamma^L > 0, \gamma^U > 0$ and $\zeta_j \geq 0, j = \overline{1, m}, \lambda_i \in \mathbb{R}, i = \overline{1, n}$, such that the functional $\gamma^L K^L + \gamma^U K^U$ is $(q, w) - \pi_1$ -pseudoinvex at (z, r) with respect to τ, ν and σ , and the functionals G, H are $(q, w) - \pi_2$ -quasiinvex and $(q, w) - \pi_3$ -quasiinvex at (z, r) with respect to τ, ν and σ , respectively, with $\pi_1 + \pi_2 + \pi_3 \geq 0$. Then

$$K(x, u) \geq_{LU} K(z, r).$$

Theorem 14. (Strong duality) If (x^*, u^*) is an LU -optimal solution to (P) , then there exist the real scalars $\gamma^{*L} > 0, \gamma^{*U} > 0, \zeta^* \geq 0$ and $\lambda^* \in \mathbb{R}$ such that $(x^*, u^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ is a feasible point for $(D-MW)$. In addition, the two objective values are equal. Moreover, if Theorem 13 holds for all feasible points $(z^*, r^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$, then $(x^*, u^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ is an LU -optimal solution of $(D-MW)$.

Theorem 15. (Strict converse duality) Consider the points (x^*, u^*) and $(z^*, r^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ are some feasible points to (P) and $(D-MW)$, respectively. Assume that the functional $\gamma^{*L} K^L + \gamma^{*U} K^U$ is strictly $(q, w) - \pi_1$ -pseudoinvex and G, H are $(q, w) - \pi_2$ -quasiinvex and $(q, w) - \pi_3$ -quasiinvex at (z^*, r^*) with respect to τ, ν and σ , with $\pi_1 + \pi_2 + \pi_3 \geq 0$ and

$$\begin{aligned} & \gamma^{*L} K^L(x^*, u^*) + \gamma^{*U} K^U(x^*, u^*) \\ & \leq \gamma^{*L} K^L(z^*, r^*) + \gamma^{*U} K^U(z^*, r^*). \end{aligned} \quad (12)$$

Then $(x^*, u^*) = (z^*, r^*)$.

The derived main theorems are new in the specialized literature. The study of such classes of interval-valued variational control problems, by considering generalized invex functionals, represents the main novelty element. The main difference between Wolfe type dual problem and Mond-Weir type dual problem, associated with the primal optimization problem (P) , is the objective (cost) functional. Specifically, in the Wolfe type dual, we consider constraint functionals in the definition of cost functional, while in the Mond-Weir type dual we keep the same cost functional as in the primal problem. On the other hand, we all know the importance of duality theory in optimization. In short, duality is basically a unifying principle that develops the relationship between two constrained optimization problems, one of them known as the primal problem, which is a problem of minimization (maximization), and the other one referred to as the dual problem, which is a problem of maximization (minimization). In the duality theory, two problems are associated in such a way that the existence of a solution to one problem ensures the solution of the other one under certain assumptions.

The approach to variational control problems governed by invariant convex interval-valued functionals represents an active area of research in many directions. One of this is well-posedness under invariant convex interval-valued functionals. This represents an open research problem because a full well-posedness theory has not yet been developed.

The content of this chapter is based on the paper *On optimal control problems with generalized invariant convex interval-valued functionals*, authored by Treanță, S., Ciontescu, M., published in *Journal of Industrial and Management Optimization*, Vol. 20, No. 11, 3317-3336, 2024; DOI: 10.3934/jimo.2024055; WOS: 001214796900001; ISSN: 1547-5816 [117].

In Chapter 3, titled *Multiple objective interval-valued optimization models under controlled dynamical system constraints*, we study a multi-dimensional interval-valued controlled optimiza-

tion model with controlled dynamical system constraints for the vector case and we construct an equivalent unconstrained optimization model using penalty function method.

We consider the following multiple objective interval-valued optimization model with controlled dynamical system constraints as:

$$\begin{aligned}
 \text{(P)} \quad & \min_{(p(\cdot), q(\cdot))} \int_{\Omega_{s_0, s_1}} \mathcal{F}(s, p(s), q(s)) dw = \left(\int_{\Omega_{s_0, s_1}} f_1(s, p(s), q(s)) dw, \dots, \right. \\
 & \left. \int_{\Omega_{s_0, s_1}} f_k(s, p(s), q(s)) dw \right) \\
 \text{subject to} \quad & \mathfrak{g}_\eta(s, p(s), q(s)) \leq 0, \quad \eta = \overline{1, r}, \\
 & \frac{\partial p^\alpha}{\partial s^u} = H_u^\alpha(s, p(s), q(s)), \quad \alpha = \overline{1, m}, \quad u \in \overline{1, l}, \\
 & p(s_0) = p_0 = \text{given}, p(s_1) = p_1 = \text{given},
 \end{aligned}$$

where $s \in \Omega_{s_0, s_1}$ and each $f_\beta, \beta = \overline{1, k}$ is an interval-valued functional, that is

$$\int_{\Omega_{s_0, s_1}} f_\beta(s, p(s), q(s)) dw = \int_{\Omega_{s_0, s_1}} \left[f_\beta^L(s, p(s), q(s)), f_\beta^U(s, p(s), q(s)) \right] dw, \beta = \overline{1, k}.$$

The functionals $f_\beta^L = (f_\beta^L) : \Omega_{s_0, s_1} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^k, f_\beta^U = (f_\beta^U) : \Omega_{s_0, s_1} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^k, \mathfrak{g} : \Omega_{s_0, s_1} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^r, H : \Omega_{s_0, s_1} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^{ml}$, are assumed to be continuously differentiable functionals.

Let

$$\begin{aligned}
 D = \left\{ (p(s), q(s)) \in X \times C : \mathfrak{g}_\eta(s, p(s), q(s)) \leq 0, \frac{\partial p^\alpha}{\partial s^u} = H_u^\alpha(s, p(s), q(s)), p(s_0) = p_0, \right. \\
 \left. p(s_1) = p_1, s \in \Omega_{s_0, s_1}, \eta = \overline{1, r}, \alpha = \overline{1, m}, u \in \overline{1, l} \right\}
 \end{aligned}$$

be the collection of all feasible solutions of (P).

We define an LU-weak efficient solution to (P) and the convexity for a real-valued functional and an interval-valued functional.

Definition 4. A point $(\bar{p}, \bar{q}) \in D$ is said to be an LU-weak efficient solution to (P), if for all $(p, q) \in D$, the following inequality

$$\int_{\Omega} f_\beta(\bar{\tau}) dw <_{LU} \int_{\Omega} f_\beta(\tau) dw, \beta = \overline{1, k}$$

holds.

Definition 5. A real-valued functional $\int_{\Omega} f_\beta(\tau) dw$ is said to be convex at $(\bar{p}, \bar{q})$ on $X \times C$, if the following inequality

$$\int_{\Omega} \left\{ f_\beta(\tau) - f_\beta(\bar{\tau}) \right\} dw \geq \int_{\Omega} (p - \bar{p}) \frac{\partial f_\beta}{\partial p^\alpha}(\bar{\tau}) dw + \int_{\Omega} (q - \bar{q}) \frac{\partial f_\beta}{\partial q^\kappa}(\bar{\tau}) dw$$

holds for all $(p, q) \in X \times C$.

For the multiple objective interval-valued optimization model with controlled dynamical system constraints (P), we formulate the associated lower and upper multiple objective interval-valued optimization models $(P_{(\bar{p}, \bar{q})})^L$ and $(P_{(\bar{p}, \bar{q})})^U$, respectively, at the given point $(\bar{p}, \bar{q})$, for $\beta = \overline{1, k}$, as follows:

$$\begin{aligned} (P_{(\bar{p}, \bar{q})})^L \quad & \min_{(p(\cdot), q(\cdot))} \int_{\Omega} f_{\beta}^L(s, p(s), q(s)) dw \\ \text{subject to} \quad & \int_{\Omega} f_{\beta}^U(s, p(s), q(s)) dw \leq \int_{\Omega} f_{\beta}^U(s, \bar{p}(s), \bar{q}(s)) dw, \\ & \mathbf{g}_{\eta}(s, p(s), q(s)) \leq 0, \quad \eta = \overline{1, r}, s \in \Omega, \\ & \frac{\partial p^{\alpha}}{\partial s^u} = H_u^{\alpha}(s, p(s), q(s)), \quad \alpha = \overline{1, m}, u \in \overline{1, l}, \\ & p(s_0) = p_0, p(s_1) = p_1, \end{aligned}$$

and

$$\begin{aligned} (P_{(\bar{p}, \bar{q})})^U \quad & \min_{(p(\cdot), q(\cdot))} \int_{\Omega} f_{\beta}^U(s, p(s), q(s)) dw \\ \text{subject to} \quad & \int_{\Omega} f_{\beta}^L(s, p(s), q(s)) dw \leq \int_{\Omega} f_{\beta}^L(s, \bar{p}(s), \bar{q}(s)) dw, \\ & \mathbf{g}_{\eta}(s, p(s), q(s)) \leq 0, \quad \eta = \overline{1, r}, s \in \Omega, \\ & \frac{\partial p^{\alpha}}{\partial s^u} = H_u^{\alpha}(s, p(s), q(s)), \quad \alpha = \overline{1, m}, u \in \overline{1, l}, \\ & p(s_0) = p_0, p(s_1) = p_1. \end{aligned}$$

We prove the equivalence between the LU-weak efficient solution to (P) and the LU-weak efficient solution to $(P_{(\bar{p}, \bar{q})})^L$ and $(P_{(\bar{p}, \bar{q})})^U$.

We discuss the necessary efficiency conditions and establish sufficient efficiency conditions to LU-weak efficient solution for (P).

Theorem 16. (Necessary Efficiency Conditions) *Under the constraint conditions (for the existence of multipliers), if $(\bar{p}, \bar{q}) \in D$ is an LU-weak efficient solution to (P), then there exist piecewise smooth Lagrange multipliers $\bar{\lambda}_{\beta}^L, \bar{\lambda}_{\beta}^U \in \mathbb{R}, \beta = \overline{1, k}, \bar{\mu}_{\eta} \in \mathbb{R}, \eta = \overline{1, r}$, and $\bar{\gamma}_u^{\alpha} \in \mathbb{R}, \alpha = \overline{1, m}, u = \overline{1, l}$, such that the following conditions (with summation over the repeated indices)*

$$\bar{\lambda}_{\beta}^L \frac{\partial f_{\beta}^L}{\partial p^{\alpha}}(\bar{\tau}) + \bar{\lambda}_{\beta}^U \frac{\partial f_{\beta}^U}{\partial p^{\alpha}}(\bar{\tau}) + \bar{\mu}_{\eta} \frac{\partial \mathbf{g}_{\eta}}{\partial p^{\alpha}}(\bar{\tau}) + \bar{\gamma}_u^{\alpha} \frac{\partial H_u^{\alpha}}{\partial p^{\alpha}}(\bar{\tau}) + \frac{\partial \bar{\gamma}_u^{\alpha}}{\partial s^u} = 0, \quad \alpha = \overline{1, m}, \quad (13)$$

$$\bar{\lambda}_{\beta}^L \frac{\partial f_{\beta}^L}{\partial q^{\kappa}}(\bar{\tau}) + \bar{\lambda}_{\beta}^U \frac{\partial f_{\beta}^U}{\partial q^{\kappa}}(\bar{\tau}) + \bar{\mu}_{\eta} \frac{\partial \mathbf{g}_{\eta}}{\partial q^{\kappa}}(\bar{\tau}) + \bar{\gamma}_u^{\alpha} \frac{\partial H_u^{\alpha}}{\partial q^{\kappa}}(\bar{\tau}) = 0, \quad \kappa = \overline{1, n}, \quad (14)$$

$$\bar{\mu}_{\eta} \mathbf{g}_{\eta}(\bar{\tau}) = 0, \quad (15)$$

$$(\bar{\lambda}_{\beta}^L, \bar{\lambda}_{\beta}^U) \geq 0, \bar{\mu}_{\eta} \geq 0 \quad (16)$$

hold for all $s \in \Omega$, except at discontinuities.

Theorem 17. (Sufficient Efficiency Conditions) *Let $(\bar{p}, \bar{q}) \in D$ and there exist the piecewise smooth Lagrange multipliers $\bar{\lambda}_{\beta}^L, \bar{\lambda}_{\beta}^U \in \mathbb{R}, \beta = \overline{1, k}, \bar{\mu}_{\eta} \in \mathbb{R}, \eta = \overline{1, r}$, and $\bar{\gamma}_u^{\alpha} \in \mathbb{R}, \alpha =$*

$\overline{1, m, u} = \overline{1, l}$, such that the necessary efficiency conditions (13)-(16) hold at $(\bar{p}, \bar{q})$ for all $s \in \Omega$, except at discontinuities. Further, assume that $\int_{\Omega} \langle \bar{\lambda}_{\beta}^L, f_{\beta}^L(\tau) \rangle dw, \int_{\Omega} \langle \bar{\lambda}_{\beta}^U, f_{\beta}^U(\tau) \rangle dw, \beta = \overline{1, k}, \int_{\Omega} \langle \bar{\mu}_{\eta}, \mathbf{g}_{\eta}(\tau) \rangle dw, \eta = \overline{1, r}$ and $\int_{\Omega} \langle \bar{\gamma}_u, H_u(\tau) - \frac{\partial p}{\partial s^u} \rangle dw, u = \overline{1, l}$ are convex at $(\bar{p}, \bar{q})$ on D . Then, $(\bar{p}, \bar{q})$ is an LU-weak efficient solution to (P).

We construct the following multiple objective unconstrained interval-valued optimization model $(P)_{\varrho}$, using the l_1 penalty function method, associated with the problem (P) as:

$$(P)_{\varrho} \min_{(p(\cdot), q(\cdot))} \mathcal{P}((p, q), \varrho) = \int_{\Omega_{s_0, s_1}} \left\{ \mathcal{F}(s, p(s), q(s)) + \varrho(s) \left[\sum_{\eta=1}^r \mathbf{g}_{\eta}^{+}(s, p(s), q(s)) + \sum_{\alpha=1}^m \sum_{u=1}^l \left| H_u^{\alpha}(s, p(s), q(s)) - \frac{\partial p^{\alpha}}{\partial s^u} \right| \right] e \right\} dw,$$

where $e = (1, \dots, 1) \in \mathbb{R}^k$, the penalty parameter $\varrho(s) > 0$ and penalty function $\mathbf{g}_{\eta}^{+}(s, p(s), q(s))$ is defined as

$$\mathbf{g}_{\eta}^{+}(s, p(s), q(s)) = \begin{cases} 0, & \text{if } \mathbf{g}_{\eta}(s, p(s), q(s)) \leq 0, \\ \mathbf{g}_{\eta}(s, p(s), q(s)), & \text{if } \mathbf{g}_{\eta}(s, p(s), q(s)) > 0. \end{cases} \quad (17)$$

Equivalently,

$$(P)_{\varrho} \min_{(p(\cdot), q(\cdot))} \mathcal{P}((p, q), \varrho) = \left(\int_{\Omega_{s_0, s_1}} f_{\beta}^L(\tau) + \varrho \left(\sum_{\eta=1}^r \mathbf{g}_{\eta}^{+}(\tau) + \sum_{\alpha=1}^m \sum_{u=1}^l \left| H_u^{\alpha}(\tau) - \frac{\partial p^{\alpha}}{\partial s^u} \right| \right) dw, \int_{\Omega_{s_0, s_1}} f_{\beta}^U(\tau) + \varrho \left(\sum_{\eta=1}^r \mathbf{g}_{\eta}^{+}(\tau) + \sum_{\alpha=1}^m \sum_{u=1}^l \left| H_u^{\alpha}(\tau) - \frac{\partial p^{\alpha}}{\partial s^u} \right| \right) dw \right)_{\beta=\overline{1, k}}.$$

We define an LU-weak efficient solution to $(P)_{\varrho}$:

Definition 6. A point $(\bar{p}, \bar{q}) \in X \times C$ is said to be an LU-weak efficient solution to $(P)_{\varrho}$, if for all $(p, q) \in X \times C$ the following inequality

$$\mathcal{P}((\bar{p}, \bar{q}), \varrho) <_{LU} \mathcal{P}((p, q), \varrho)$$

holds.

We demonstrate the equivalence between the original system and the unconstrained system through two theorems.

Theorem 18. Let $(\bar{p}, \bar{q}) \in D$ and the necessary efficiency conditions (13)-(16) be satisfied at $(\bar{p}, \bar{q})$ with $\sum_{\beta=1}^k \bar{\lambda}_{\beta}^L + \sum_{\beta=1}^k \bar{\lambda}_{\beta}^U = 1$. Further, assume that $\int_{\Omega} \langle \bar{\lambda}_{\beta}^L, f_{\beta}^L(\tau) \rangle dw, \int_{\Omega} \langle \bar{\lambda}_{\beta}^U, f_{\beta}^U(\tau) \rangle dw, \beta = \overline{1, k}, \int_{\Omega} \langle \bar{\mu}_{\eta}, \mathbf{g}_{\eta}(\tau) \rangle dw, \eta = \overline{1, r}$ and $\int_{\Omega} \langle \bar{\gamma}_u, H_u(\tau) - \frac{\partial p^{\alpha}}{\partial s^u} \rangle dw, \alpha = \overline{1, m}, u = \overline{1, l}$ are convex at $(\bar{p}, \bar{q})$ on $X \times C$. Furthermore, if we assume that $\varrho \geq \max\{\bar{\mu}_{\eta}, \eta = \overline{1, r}, |\bar{\gamma}_u^{\alpha}|, \alpha = \overline{1, m}, u = \overline{1, l}\}$, then $(\bar{p}, \bar{q})$ is an LU-weak efficient solution to $(P)_{\varrho}$.

Now, we shall prove the converse result of Theorem 18.

Theorem 19. Let $(\bar{p}, \bar{q})$ be an LU-weak efficient solution to $(P)_{\varrho}$ and the feasible set D be a compact subset of $X \times C$. Let $\varrho \geq \max\{\bar{\mu}_{\eta}, \eta = \overline{1, r}, |\bar{\gamma}_u^{\alpha}|, \alpha = \overline{1, m}, u = \overline{1, l}\}$. Further, assume that, $\int_{\Omega} \langle \bar{\lambda}_{\beta}^L, f_{\beta}^L(\tau) \rangle dw, \int_{\Omega} \langle \bar{\lambda}_{\beta}^U, f_{\beta}^U(\tau) \rangle dw, \beta = \overline{1, k}, \int_{\Omega} \langle \bar{\mu}_{\eta}, \mathbf{g}_{\eta}(\tau) \rangle dw, \eta = \overline{1, r}$ and

$\int_{\Omega} \langle \bar{\gamma}_u^\alpha, H_u^\alpha(\tau) - \frac{\partial p^\alpha}{\partial s^u} \rangle dw, \alpha = \overline{1, m}$ are convex on $X \times C$. Then, $(\bar{p}, \bar{q})$ is an LU-weak efficient solution to (P).

We formulate an example to authenticate the established result.

As further research direction, it would be interesting to investigate similar results for fuzzy-valued control models under dynamic system constraints with the help of the exact minimax penalty function method.

The content of this chapter is based on the paper *On multiple objective interval-valued optimization models with controlled dynamical system constraints*, authored by Preeti, Treanță, S., Ciontescu, M., published in *International Journal of Control*, Vol. 98, No. 7, pp. 1713-1721, 2025; DOI: 10.1080/00207179.2024.2423212; WOS: 001348217100001; ISSN: 0020-7179 [91].

The aim of **Chapter 4**, titled *Characterization and existence results of solutions in non-convex controlled minimization models*, is to provide some characterization and existence results of solutions in non-convex controlled multiobjective minimization models governed by invexity.

Consider $\Theta : B \times C \rightarrow \mathbb{R}$ is defined as $\Theta(p, n) = \int_{\Omega_{v_1, v_2}} f(v, p, p_\zeta, n) dv$, where $f : \Omega_{v_1, v_2} \times \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^\omega \rightarrow \mathbb{R}$ is a continuously differentiable multiple integral type functional (see $dv := dv^1 \cdots dv^q$ as volume element in $\mathbb{R}^q$). Next, we establish the (generalized) (χ, ρ) -invexity associated with Θ .

This study focuses on efficiency criteria and the associated duals for the following non-convex multi-cost extremization problem:

$$\begin{aligned} \text{(NMEP)} \quad & \min_{(p, n)} \int_{\Omega_{v_1, v_2}} \Upsilon(v, p(v), n(v)) dv \\ & = \left(\int_{\Omega_{v_1, v_2}} \Upsilon^1(v, p(v), n(v)) dv, \dots, \int_{\Omega_{v_1, v_2}} \Upsilon^z(v, p(v), n(v)) dv \right) \\ & \text{subject to} \quad U(v, p(v), n(v)) \leq 0, \quad v \in \Omega_{v_1, v_2}, \\ & \quad Z(v, p(v), n(v)) = p_\zeta(v), \quad v \in \Omega_{v_1, v_2}, \\ & \quad p(v_1) = p_1 = \text{given}, \quad p(v_2) = p_2 = \text{given}, \end{aligned}$$

where $\Upsilon = (\Upsilon^1, \dots, \Upsilon^z) : \Omega_{v_1, v_2} \times \mathbb{R}^r \times \mathbb{R}^\omega \rightarrow \mathbb{R}^z$ is a z -dimensional C^1 -class functional, and the constraint functionals $U : \Omega_{v_1, v_2} \times \mathbb{R}^r \times \mathbb{R}^\omega \rightarrow \mathbb{R}^x$ and $Z : \Omega_{v_1, v_2} \times \mathbb{R}^r \times \mathbb{R}^\omega \rightarrow \mathbb{R}^r$ are assumed to be continuously differentiable x -dimensional and r -dimensional functionals, respectively.

Let V be the feasible solution set for (NMEP), that is

$$V = \{(p, n) \in B \times C \text{ satisfying the mentioned constraints in (NMEP)}\}.$$

We define an *efficient point* of (NMEP).

Definition 7. A pair $(\bar{p}, \bar{n}) \in V$ associated to (NMEP) is an *efficient point* of (NMEP) if there exists no other $(p, n) \in V$ such that

$$\int_{\Omega_{v_1, v_2}} \Upsilon(v, p, n) dv \leq \int_{\Omega_{v_1, v_2}} \Upsilon(v, \bar{p}, \bar{n}) dv$$

that is, there exists no other $(p, n) \in V$ such that

$$\begin{aligned} \int_{\Omega_{v_1, v_2}} \Upsilon^\iota(v, p, n) dv &\leq \int_{\Omega_{v_1, v_2}} \Upsilon^\iota(v, \bar{p}, \bar{n}) dv, \quad \forall \iota \in H, \\ \int_{\Omega_{v_1, v_2}} \Upsilon^s(v, p, n) dv &< \int_{\Omega_{v_1, v_2}} \Upsilon^s(v, \bar{p}, \bar{n}) dv, \quad \text{for some } s \in H. \end{aligned}$$

We formulate the KKT-type efficiency conditions (necessary criterion) associated with our problem, and we present and prove several theorems that provide the sufficiency of the KKT conditions, under various (χ, ϱ) -invexity, pseudoinvexity and quasi-invexity hypotheses.

Theorem 20. *Let $(\bar{p}, \bar{n})$ be a normal efficient point for (NMEP) and the KKT constraint qualification be satisfied. Then there are $\bar{\eta} \in \mathbb{R}^z$, $\bar{\lambda}(\cdot) : \Omega_{v_1, v_2} \rightarrow \mathbb{R}^x$ and $\bar{\pi}(\cdot) : \Omega_{v_1, v_2} \rightarrow \mathbb{R}^r$ satisfying*

$$\begin{aligned} &\bar{\eta}^T \Upsilon_p(v, \bar{p}, \bar{n}) + \bar{\lambda}(v)^T G_p(v, \bar{p}, \bar{n}) + \bar{\pi}(v)^T H_p(v, \bar{p}, \bar{p}_\zeta, \bar{n}) \\ &= \frac{\partial}{\partial v_\zeta} [\bar{\eta}^T \Upsilon_{p_\zeta}(v, \bar{p}, \bar{n}) + \bar{\lambda}(v)^T G_{p_\zeta}(v, \bar{p}, \bar{n}) + \bar{\pi}(v)^T H_{p_\zeta}(v, \bar{p}, \bar{p}_\zeta, \bar{n})], \end{aligned} \quad (18)$$

$$\bar{\eta}^T \Upsilon_n(v, \bar{p}, \bar{n}) + \bar{\lambda}(v)^T G_n(v, \bar{p}, \bar{n}) + \bar{\pi}(v)^T H_n(v, \bar{p}, \bar{p}_\zeta, \bar{n}) = 0, \quad v \in \Omega_{v_1, v_2}, \quad (19)$$

$$\int_{\Omega_{v_1, v_2}} \bar{\lambda}(v)^T U(v, \bar{p}, \bar{n}) dv = 0, \quad \bar{\eta} \geq 0, \quad \bar{\eta}^T e = 1, \quad \bar{\lambda}(v) \geq 0, \quad v \in \Omega_{v_1, v_2}, \quad (20)$$

except at discontinuities, with $S := Z - p_\zeta$.

Theorem 21. *If $(\bar{p}, \bar{n}) \in V$, the KKT criteria in (18)-(20) are fulfilled at this pair, with $\bar{\eta} \in \mathbb{R}^z$ and $\bar{\lambda}(\cdot) : \Omega_{v_1, v_2} \rightarrow \mathbb{R}^x$ and $\bar{\pi}(\cdot) : \Omega_{v_1, v_2} \rightarrow \mathbb{R}^r$, and the hypotheses are valid:*

- (a) $\int_{\Omega_{v_1, v_2}} \Upsilon^\iota(v, p, n) dv, \iota \in H$, is strictly $(\chi, \varrho_{\Upsilon^\iota})$ -invex at $(\bar{p}, \bar{n})$ on V ;
 - (b) $\int_{\Omega_{v_1, v_2}} U^l(v, p, n) dv, l \in E$, is (χ, ϱ_{U^l}) -invex at $(\bar{p}, \bar{n})$ on V ;
 - (c) $\int_{\Omega_{v_1, v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^+(v) = \{j \in A : \bar{\pi}_j(v) > 0\}$, is (χ, ϱ_{S^j}) -invex at $(\bar{p}, \bar{n})$ on V ;
 - (d) $-\int_{\Omega_{v_1, v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^-(v) = \{j \in A : \bar{\pi}_j(v) < 0\}$, is (χ, ϱ_{S^j}) -invex at $(\bar{p}, \bar{n})$ on V ;
 - (e) $\sum_{\iota=1}^z \bar{\eta}_\iota \varrho_{\Upsilon^\iota} + \sum_{l=1}^x \bar{\lambda}_l \varrho_{U^l} + \sum_{j \in A^+(v)} \bar{\pi}_j \varrho_{S^j} - \sum_{j \in A^-(v)} \bar{\pi}_j \varrho_{S^j} \geq 0$,
- then $(\bar{p}, \bar{n})$ is an efficient point of (NMEP).

Theorem 22. *If $(\bar{p}, \bar{n}) \in V$, the KKT criteria in (18)-(20) are satisfied at this pair, with $\bar{\eta} \in \mathbb{R}^z$ and $\bar{\lambda}(\cdot) : \Omega_{v_1, v_2} \rightarrow \mathbb{R}^x$ and $\bar{\pi}(\cdot) : \Omega_{v_1, v_2} \rightarrow \mathbb{R}^r$, and the statements are fulfilled:*

- (a) $\int_{\Omega_{v_1, v_2}} \Upsilon^\iota(v, p, n) dv, \iota \in H$, is strictly $(\chi, \varrho_{\Upsilon^\iota})$ -pseudoinvex at $(\bar{p}, \bar{n})$ on V ;
- (b) $\int_{\Omega_{v_1, v_2}} U^l(v, p, n) dv, l \in E$, is (χ, ϱ_{U^l}) -quasiinvex at $(\bar{p}, \bar{n})$ on V ;
- (c) $\int_{\Omega_{v_1, v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^+(v)$, is (χ, ϱ_{S^j}) -quasiinvex at $(\bar{p}, \bar{n})$ on V ;

- (d) $-\int_{\Omega_{v_1, v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^-(v)$, is (χ, ϱ_{S^j}) -quasiinvex at $(\bar{p}, \bar{n})$ on V ;
- (e) $\sum_{\iota=1}^z \bar{\eta}_\iota \varrho_{U^\iota} + \sum_{l=1}^x \bar{\lambda}_l \varrho_{U^l} + \sum_{j \in A^+(v)} \bar{\pi}_j \varrho_{S^j} - \sum_{j \in A^-(v)} \bar{\pi}_j \varrho_{S^j} \geq 0$,
then $(\bar{p}, \bar{n})$ is an efficient point of (NMEP).

Let M, L be some subsets of E such that $M \cup L = E$, $M \cap L = \emptyset$, and let

$$\lambda_M(v)^T U^M(v, \gamma(v), \delta(v)) := \sum_{l \in M} \lambda_l(v) U^l(v, \gamma(v), \delta(v))$$

and

$$\lambda_L(v)^T U^L(v, \gamma(v), \delta(v)) := \sum_{l \in L} \lambda_l(v) U^l(v, \gamma(v), \delta(v)).$$

We construct a dual model (DUAL), formulated as below:

$$(\text{Dual}) \quad \max_{(\gamma, \delta)} \int_{\Omega_{v_1, v_2}} (\Upsilon(v, \gamma(v), \delta(v)) + \lambda_M(v)^T U^M(v, \gamma(v), \delta(v)) e) dv$$

subject to

$$\begin{aligned} & \eta^T \Upsilon_p(v, \gamma(v), \delta(v)) + \lambda(v)^T G_p(v, \gamma(v), \delta(v)) + \pi(v)^T H_p(v, \gamma(v), \dot{\gamma}(v), \delta(v)) \\ &= \frac{\partial}{\partial v^\zeta} \left[\eta^T \Upsilon_{p_\zeta}(v, \gamma(v), \delta(v)) + \lambda(v)^T G_{p_\zeta}(v, \gamma(v), \delta(v)) \right. \\ & \quad \left. + \pi(v)^T H_{p_\zeta}(v, \gamma(v), \dot{\gamma}(v), \delta(v)) \right], \\ & \eta^T \Upsilon_n(v, \gamma(v), \delta(v)) + \lambda(v)^T G_n(v, \gamma(v), \delta(v)) + \pi(v)^T H_n(v, \gamma(v), \dot{\gamma}(v), \delta(v)) = 0, \\ & \int_{\Omega_{v_1, v_2}} \lambda_L(v)^T U^L(v, \gamma(v), \delta(v)) dv \geq 0, \\ & \int_{\Omega_{v_1, v_2}} \pi(v)^T S(v, \gamma(v), \dot{\gamma}(v), \delta(v)) dv = 0, \\ & \gamma(v_1) = p_1, \gamma(v_2) = p_2, \eta \geq 0, \eta^T e = 1, \lambda(v) \geq 0, v \in \Omega_{v_1, v_2}, \end{aligned}$$

where $e = (1, 1, \dots, 1) \in \mathbb{R}^z$ is a z -dimensional vector.

We prove a duality connection between (NMEP) and its mixed multi-cost dual model (Dual).

Theorem 23. Let (p, n) and $(\gamma, \delta, \eta, \lambda, \pi)$ be any feasible solutions of (NMEP) and (Dual), respectively. In addition, we consider the statements are fulfilled:

- (a) $\int_{\Omega_{v_1, v_2}} \Upsilon^\iota(v, p, n) dv, \iota \in H$, is strictly $(\chi, \varrho_{\Upsilon^\iota})$ -invex at (γ, δ) on $V \cup Y$;
- (n) $\int_{\Omega_{v_1, v_2}} U^l(v, p, n) dv, l \in E$, is (χ, ϱ_{U^l}) -invex at (γ, δ) on $V \cup Y$;
- (c) $\int_{\Omega_{v_1, v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^+(v)$, is (χ, ϱ_{S^j}) -invex at (γ, δ) on $V \cup Y$;

(d) $-\int_{\Omega_{v_1, v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^-(v)$, is (χ, ϱ_{S^j}) -invex at (γ, δ) on $V \cup Y$;

(e) $\sum_{\iota=1}^z \eta_\iota \varrho_{\Upsilon^\iota} + \sum_{l=1}^x \lambda_l \varrho_{U^l} + \sum_{j \in A^+(v)} \kappa_j \varrho_{S^j} - \sum_{j \in A^-(v)} \kappa_j \varrho_{S^j} \geq 0$.

Then, the relations cannot hold

$$\int_{\Omega_{v_1, v_2}} \Upsilon^\iota(v, p, n) dv \leq \int_{\Omega_{v_1, v_2}} (\Upsilon^\iota(v, \gamma, \delta) + \lambda_M(v)^T U^M(v, \gamma, \delta)) dv, \quad \iota \in H, \quad (21)$$

and

$$\int_{\Omega_{v_1, v_2}} \Upsilon^s(v, p, n) dv < \int_{\Omega_{v_1, v_2}} (\Upsilon^s(v, \gamma, \delta) + \lambda_M(v)^T U^M(v, \gamma, \delta)) dv, \quad (22)$$

for some $s \in H$.

The results of this chapter are new in the specialized literature. The main contribution is given by the multi-dimensional framework in which the problem is built. More specifically, the presence of multiple integrals as cost functionals in the studied optimization problem. This implies updated definitions for the concept of (χ, ϱ) -invexity (and its variants) associated with functionals governed by multiple integrals and not simple integrals, as usual.

Most algorithms for non-convex optimization only guarantee convergence to stationary points and completely ignore the generalized invexity structure.

As a future direction, we mention the extension to algorithms that exploit generalized invexity and convergence to efficient solutions (not just stationary).

The content of this chapter is based on the paper *Characterization and existence results of solutions in non-convex controlled minimization models*, authored by Treanță, S., Ciontescu, M., Laha, V., Shi, F., published in *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, Vol. 17, No. 1, 67-94, 2025; DOI: 10.56082/annalsarscimath.2025.1.67; ISSN: 2066-6594 [119].

In **Chapter 5**, titled *On multiple-cost optimization and extended controlled vector inequalities*, we improve the study of vector variational inequalities, in the Minty and Stampacchia forms. More precisely, we incorporate a new variable to these classes of variational problems, namely, the control function. Also, in this new framework, we present the notions of preconvexity and (strictly) strong convexity for functionals driven by controlled simple integrals, and a result of mean value type.

We state the following multiobjective minimization model:

$$(P) \quad \min_{(\omega, \theta)} \int_{\mathcal{Y}} \psi(t, \omega, \theta) dt = \left(\int_{\mathcal{Y}} \psi^1(t, \omega, \theta) dt, \dots, \int_{\mathcal{Y}} \psi^r(t, \omega, \theta) dt \right),$$

subject to

$$\omega(a) = v, \quad \omega(b) = \chi,$$

$$\pi(t, \omega, \theta) \leq 0, \quad t \in \mathcal{Y},$$

$$H(t, \omega, \dot{\omega}, \theta) := \dot{\omega} - h(t, \omega, \theta) = 0, \quad t \in \mathcal{Y},$$

where $\pi : \mathcal{Y} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k$ and $h : \mathcal{Y} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are considered to be continuous differentiable functionals. Let $\Omega \subset X \times U$ be the convex set of feasible points of P .

We define an efficient point and a weak efficient point of P .

Definition 8. We say $(\vartheta, \mu) \in \Omega$ is an efficient point of P if the inequality

$$\int_{\mathcal{Y}} \psi^\kappa(t, \omega, \theta) dt \leq \int_{\mathcal{Y}} \psi^\kappa(t, \vartheta, \mu) dt$$

is not satisfied, for all $(\omega, \theta) \in \Omega$, with $<$ for at least one $\kappa \in \mathcal{B}$, with $\mathcal{B} = \{1, \dots, r\}$.

Definition 9. We say $(\vartheta, \mu) \in \Omega$ is a weak efficient point of P if the inequality

$$\int_{\mathcal{Y}} \psi^\kappa(t, \omega, \theta) dt < \int_{\mathcal{Y}} \psi^\kappa(t, \vartheta, \mu) dt, \quad \forall \kappa \in \mathcal{B},$$

is not satisfied, for all $(\omega, \theta) \in \Omega$.

We present the notions of preconvexity and (strictly) strong convexity for functionals driven by controlled simple integrals, and a result of mean value type.

Definition 10. The functional of simple integral type

$$Q : \Omega \rightarrow \mathbb{R}, \quad Q(\omega, \theta) = \int_{\mathcal{Y}} q(t, \omega, \theta) dt$$

is strongly preconvex on Ω if there exists a constant $\lambda > 0$ such that, for all $(\omega, \theta), (\vartheta, \mu) \in \Omega$ and $\rho \in [0, 1]$, the following inequality holds:

$$\begin{aligned} & \int_{\mathcal{Y}} q(t, \vartheta + \rho(\omega - \vartheta), \mu + \rho(\theta - \mu)) dt \\ & \leq \rho \int_{\mathcal{Y}} q(t, \omega, \theta) dt + (1 - \rho) \int_{\mathcal{Y}} q(t, \vartheta, \mu) dt - \lambda \rho(1 - \rho) \|(\omega, \theta) - (\vartheta, \mu)\|^2. \end{aligned}$$

Definition 11. The functional of simple integral type

$$Q : \Omega \rightarrow \mathbb{R}, \quad Q(\omega, \theta) = \int_{\mathcal{Y}} q(t, \omega, \theta) dt$$

is strongly convex on Ω if there exists a constant $\lambda > 0$ such that, for all $(\omega, \theta), (\vartheta, \mu) \in \Omega$, the following inequality holds:

$$\begin{aligned} & \int_{\mathcal{Y}} [q_\omega(t, \vartheta, \mu)(\omega - \vartheta) + q_\theta(t, \vartheta, \mu)(\theta - \mu)] dt + \lambda \|(\omega, \theta) - (\vartheta, \mu)\|^2 \\ & \leq \int_{\mathcal{Y}} q(t, \omega, \theta) dt - \int_{\mathcal{Y}} q(t, \vartheta, \mu) dt. \end{aligned}$$

We formulate an example of a convex functional that is not a strong convex functional. However, the converse is always true.

Next, we state the generalized vector variational controlled inequalities of Minty and Stampacchia type:

GMVVI $_{\xi}$: For a given ξ , let us find $(\vartheta, \mu) \in \Omega$ so that $\exists$ no $(\omega, \theta) \in \Omega$, fulfilling

$$\int_{\mathcal{Y}} [\psi_{\omega}^{\kappa}(t, \omega, \theta) (\vartheta - \omega) + \psi_{\theta}^{\kappa}(t, \omega, \theta) (\mu - \theta)] dt + \xi \|(\vartheta, \mu) - (\omega, \theta)\|^2 \geq 0,$$

with $>$ for at least one $\kappa \in \mathcal{B}$;

GSVVI $_{\xi}$: For a given ξ , let us find $(\vartheta, \mu) \in \Omega$ so that $\exists$ no $(\omega, \theta) \in \Omega$, fulfilling

$$\int_{\mathcal{Y}} [\psi_{\omega}^{\kappa}(t, \vartheta, \mu) (\omega - \vartheta) + \psi_{\theta}^{\kappa}(t, \vartheta, \mu) (\theta - \mu)] dt + \xi \|(\omega, \theta) - (\vartheta, \mu)\|^2 \leq 0,$$

with $<$ for at least one $\kappa \in \mathcal{B}$;

GWMVVI $_{\xi}$: For a given ξ , let us find $(\vartheta, \mu) \in \Omega$ so that $\exists$ no $(\omega, \theta) \in \Omega$, fulfilling

$$\int_{\mathcal{Y}} [\psi_{\omega}^{\kappa}(t, \omega, \theta) (\vartheta - \omega) + \psi_{\theta}^{\kappa}(t, \omega, \theta) (\mu - \theta)] dt + \xi \|(\vartheta, \mu) - (\omega, \theta)\|^2 > 0, \forall \kappa \in \mathcal{B};$$

GWSVVI $_{\xi}$: For a given ξ , let us find $(\vartheta, \mu) \in \Omega$ so that $\exists$ no $(\omega, \theta) \in \Omega$, fulfilling

$$\int_{\mathcal{Y}} [\psi_{\omega}^{\kappa}(t, \vartheta, \mu) (\omega - \vartheta) + \psi_{\theta}^{\kappa}(t, \vartheta, \mu) (\theta - \mu)] dt + \xi \|(\omega, \theta) - (\vartheta, \mu)\|^2 < 0, \forall \kappa \in \mathcal{B}.$$

In the following, we present various connections between the given multiple-cost optimization model and the considered extended controlled vector inequalities.

Theorem 24. For each $\kappa \in \mathcal{B}$, let us assume that the functional $\Psi^{\kappa} : \Omega \rightarrow \mathbb{R}$, $\Psi^{\kappa}(\omega, \theta) = \int_{\mathcal{Y}} \psi^{\kappa}(t, \omega, \theta) dt$ is strongly convex on Ω with constant λ_{κ} . Then the pair $(\vartheta, \mu) \in \Omega$ is an efficient point of P if and only if it is a solution of **GMVVI $_{\xi}$** , where $\xi = \min\{\lambda_1, \dots, \lambda_r\}$.

We build the next numerical example to illustrate the result stated in the above-mentioned theorem.

Theorem 25. Assume that $\Psi^{\kappa} = \int_{\mathcal{Y}} \psi^{\kappa}(t, \cdot, \cdot) dt$ is strongly convex on Ω , for every $\kappa \in \mathcal{B}$, with constant λ_{κ} . If $(\vartheta, \mu) \in \Omega$ is a solution to **GSVVI $_{\xi}$** , $\xi = \min\{\lambda_1, \dots, \lambda_r\}$, then $(\vartheta, \mu) \in \Omega$ is an efficient point in P .

Theorem 26. Assume that $\Psi^{\kappa} = \int_{\mathcal{Y}} \psi^{\kappa}(t, \cdot, \cdot) dt$ is strongly convex on Ω with λ_{κ} , $\kappa \in \mathcal{B}$. If $(\vartheta, \mu) \in \Omega$ solves **GWSVVI $_{\xi}$** , then it solves **GWMVVI $_{\xi}$** , where $\xi = \min\{\lambda_1, \dots, \lambda_r\}$.

Theorem 27. Assume that, for each $\kappa \in \mathcal{B}$, the integral $\Psi^{\kappa} = \int_{\mathcal{Y}} \psi^{\kappa}(t, \cdot, \cdot) dt$ is strongly convex on Ω with respect to λ_{κ} . If (ϑ, μ) is a solution to **GWSVVI $_{\xi}$** , $\xi = \min\{\lambda_1, \dots, \lambda_r\}$, then (ϑ, μ) is a weak efficient point in P .

Theorem 28. Consider, for each $\kappa \in \mathcal{B}$, the integral $\Psi^{\kappa} = \int_{\mathcal{Y}} \psi^{\kappa}(t, \cdot, \cdot) dt$ is strictly strongly convex on Ω with respect to λ_{κ} . If (ϑ, μ) is a weak efficient point in P , then (ϑ, μ) is a solution to **GMVVI $_{\xi}$** , $\xi = \min\{\lambda_1, \dots, \lambda_r\}$.

Theorem 29. Consider $\Psi^\kappa = \int_{\mathcal{Y}} \psi^\kappa(t, \cdot, \cdot) dt$ is a strongly convex functional on Ω with respect to λ_κ , for each $\kappa \in \mathcal{B}$. If (ϑ, μ) is a weak efficient point of P , then it is a solution of GWMVVI_ξ , $\xi = \min\{\lambda_1, \dots, \lambda_r\}$.

The main results derived in this study have been formulated in 6 principal theorems. The proofs associated with these theoretical outcomes are innovative and non-trivial. Concretely, the novel elements associated with the current study are:

- introduction of generalized Minty and Stampacchia controlled (weak) vector variational inequalities;
- introduction of the corresponding multiobjective extremization models;
- establishing various connections between the considered classes of variational problems;
- adjusting the concepts of (strictly) strong convexity and preconvexity.

As future research directions, strongly connected with the topic investigated in this chapter, we can mention the study of well-posedness of such classes of extremization models, the duality theory, and saddle-point criteria associated with the considered optimization problems. Moreover, we can suggest the case of multiple or curvilinear integrals as functionals for the mentioned studies. In these cases, there are various potential applications of the proposed models in engineering, for instance, due to the physical significance of curvilinear integrals (mechanical work).

The content of this chapter is based on the paper *On multiple-cost optimization and extended controlled vector inequalities*, authored by Treanță, S., Ciontescu, M., published in *Electronic Research Archive*, Vol. 33, No. 11, pp. 7085-7102, 2025; DOI: 10.3934/era.2025313; WOS: 001626711100010; eISSN: 2688-1594 [118].

In **Chapter 6**, titled *Multiple objective variational control models governed by univex functionals*, we investigate a class of multiobjective variational control models determined by univex functionals. More specifically, we apply the technique of modified cost functional for solving the non-convex multiobjective variational control problem. As part of the methodology research, we have the following: including the control variable in the context of univex variational problems with mixed constraints; introducing the concept of univexity for controlled simple integral functionals; characterizing efficient solutions for the considered class of problems by using the new concept of univexity and the modified objective functional technique.

A mathematical formulation associated with a multiobjective variational control model is given as follows:

$$\begin{aligned}
 (P) \quad & \min_{(\varepsilon, \varsigma)} \int_{\alpha}^{\beta} \rho(n, \varepsilon(n), \varsigma(n)) dn \\
 & = \left(\int_{\alpha}^{\beta} \rho^1(n, \varepsilon(n), \varsigma(n)) dn, \dots, \int_{\alpha}^{\beta} \rho^q(n, \varepsilon(n), \varsigma(n)) dn \right) \\
 & \text{subject to} \\
 & g(n, \varepsilon(n), \varsigma(n)) \leq 0, \quad n \in K, \\
 & \dot{\varepsilon}(n) = h(n, \varepsilon(n), \varsigma(n)), \quad n \in K, \\
 & \varepsilon(\alpha) = \pi = \text{given}, \quad \varepsilon(\beta) = \sigma = \text{given},
 \end{aligned}$$

where $\rho = (\rho^1, \dots, \rho^q) : K \times \mathbb{R}^u \times \mathbb{R}^k \rightarrow \mathbb{R}^q$, $g = (g^1, \dots, g^m) : K \times \mathbb{R}^u \times \mathbb{R}^k \rightarrow \mathbb{R}^m$ and $h = (h^1, \dots, h^u) : K \times \mathbb{R}^u \times \mathbb{R}^k \rightarrow \mathbb{R}^u$ are continuously differentiable functions, and π, σ are

some given real values. Let F be the feasible solution set for the multiobjective variational control model (P) ,

$$F = \left\{ (\varepsilon, \varsigma) \in Q \times U : g(n, \varepsilon(n), \varsigma(n)) \leq 0, \dot{\varepsilon}(n) = h(n, \varepsilon(n), \varsigma(n)), \forall n \in K, \right.$$

We define an efficient point of (P) and a weakly efficient point of (P) .

Definition 12. A feasible solution $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ of the multiobjective variational control model (P) is called *weakly efficient point* of (P) if there exists no other $(\varepsilon, \varsigma) \in F$ such that

$$\int_{\alpha}^{\beta} \rho(n, \varepsilon, \varsigma) dn < \int_{\alpha}^{\beta} \rho(n, \bar{\varepsilon}, \bar{\varsigma}) dn$$

that is, there exists no $(\varepsilon, \varsigma) \in F$ satisfying

$$\int_{\alpha}^{\beta} \rho^j(n, \varepsilon, \varsigma) dn < \int_{\alpha}^{\beta} \rho^j(n, \bar{\varepsilon}, \bar{\varsigma}) dn, \quad \forall j \in M.$$

Definition 13. A feasible solution $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ of the multiobjective variational control model (P) is called *efficient point* of (P) if there exists no other $(\varepsilon, \varsigma) \in F$ such that

$$\int_{\alpha}^{\beta} \rho(n, \varepsilon, \varsigma) dn \leq \int_{\alpha}^{\beta} \rho(n, \bar{\varepsilon}, \bar{\varsigma}) dn$$

that is, there exists no $(\varepsilon, \varsigma) \in F$ satisfying

$$\begin{aligned} \int_{\alpha}^{\beta} \rho^j(n, \varepsilon, \varsigma) dn &\leq \int_{\alpha}^{\beta} \rho^j(n, \bar{\varepsilon}, \bar{\varsigma}) dn, \quad \forall j \in M, \\ \int_{\alpha}^{\beta} \rho^{j^*}(n, \varepsilon, \varsigma) dn &< \int_{\alpha}^{\beta} \rho^{j^*}(n, \bar{\varepsilon}, \bar{\varsigma}) dn, \text{ for at least one } j^* \in M. \end{aligned}$$

To state the sufficient efficiency conditions for the previously considered multiobjective variational control model (P) , we formulate the necessary efficiency conditions of Karush-Kuhn-Tucker type.

Theorem 30. If the pair $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is a normal efficient point of the multiobjective variational control model (P) , and the Karush-Kuhn-Tucker constraint qualification is fulfilled, then there exist the multipliers $\bar{\gamma} \in \mathbb{R}^q$ and $\bar{\xi} : K \rightarrow \mathbb{R}^m$ and $\bar{\nu} : K \rightarrow \mathbb{R}^u$ such that:

$$\begin{aligned} &\bar{\gamma}^T \rho_{\varepsilon}(n, \bar{\varepsilon}, \bar{\varsigma}) + \bar{\xi}(n)^T g_{\varepsilon}(n, \bar{\varepsilon}, \bar{\varsigma}) + \bar{\nu}(n)^T h_{\varepsilon}(n, \bar{\varepsilon}, \bar{\varsigma}) \\ &= \frac{d}{dn} [\bar{\gamma}^T \rho_{\dot{\varepsilon}}(n, \bar{\varepsilon}, \bar{\varsigma}) + \bar{\xi}(n)^T g_{\dot{\varepsilon}}(n, \bar{\varepsilon}, \bar{\varsigma}) - \bar{\nu}(n)^T], \\ &\bar{\gamma}^T \rho_{\varsigma}(n, \bar{\varepsilon}, \bar{\varsigma}) + \bar{\xi}(n)^T g_{\varsigma}(n, \bar{\varepsilon}, \bar{\varsigma}) + \bar{\nu}(n)^T h_{\varsigma}(n, \bar{\varepsilon}, \bar{\varsigma}) = 0, \\ &\bar{\xi}(n)^T g(n, \bar{\varepsilon}, \bar{\varsigma}) = 0, \quad \bar{\gamma} \geq 0, \quad \bar{\gamma}^T e = 1, \quad \bar{\xi}(n) \geq 0, \end{aligned}$$

are satisfied, for all $n \in K$ except at discontinuities, where $e = (1, \dots, 1) \in \mathbb{R}^q$.

We apply the technique of modified cost functional to analyze the considered optimization model for solving the non-convex multiobjective variational control model (P) . Thus, at the given point $(\bar{\varepsilon}, \bar{\varsigma}) \in F$, we introduce the new multiobjective variational control model $(P_{\kappa, \mu}(\bar{\varepsilon}, \bar{\varsigma}))$ by considering a modified cost functional associated with the initial multiple objective variational control model (P) , as:

$$\begin{aligned} (P_{\kappa, \mu}(\bar{\varepsilon}, \bar{\varsigma})) \quad & \min_{(\varepsilon, \varsigma)} \left(\int_{\alpha}^{\beta} \left\{ \kappa(n, \varepsilon, \varsigma, \bar{\varepsilon}, \bar{\varsigma})^T \rho_{\bar{\varepsilon}}^1(n, \bar{\varepsilon}, \bar{\varsigma}) \right. \right. \\ & + \mu(n, \varepsilon, \varsigma, \bar{\varepsilon}, \bar{\varsigma})^T \rho_{\bar{\varsigma}}^1(n, \bar{\varepsilon}, \bar{\varsigma}) + \frac{d}{dn} \left(\kappa(n, \varepsilon, \varsigma, \bar{\varepsilon}, \bar{\varsigma})^T \right) \rho_{\bar{\varepsilon}}^1(n, \bar{\varepsilon}, \bar{\varsigma}) \Big\} dn, \\ & \dots, \int_{\alpha}^{\beta} \left\{ \kappa(n, \varepsilon, \varsigma, \bar{\varepsilon}, \bar{\varsigma})^T \rho_{\bar{\varepsilon}}^q(n, \bar{\varepsilon}, \bar{\varsigma}) \right. \\ & \left. + \mu(n, \varepsilon, \varsigma, \bar{\varepsilon}, \bar{\varsigma})^T \rho_{\bar{\varsigma}}^q(n, \bar{\varepsilon}, \bar{\varsigma}) + \frac{d}{dn} \left(\kappa(n, \varepsilon, \varsigma, \bar{\varepsilon}, \bar{\varsigma})^T \right) \rho_{\bar{\varepsilon}}^q(n, \bar{\varepsilon}, \bar{\varsigma}) \right\} dn \Big) \end{aligned}$$

subject to

$$g(n, \varepsilon, \varsigma) \leq 0, \quad n \in K,$$

$$\dot{\varepsilon} = h(n, \varepsilon, \varsigma), \quad n \in K,$$

$$\varepsilon(\alpha) = \pi, \quad \varepsilon(\beta) = \sigma,$$

where $\rho = (\rho^1, \dots, \rho^q) : K \times \mathbb{R}^u \times \mathbb{R}^k \rightarrow \mathbb{R}^q$, $g = (g^1, \dots, g^m) : K \times \mathbb{R}^u \times \mathbb{R}^k \rightarrow \mathbb{R}^m$ and $h = (h^1, \dots, h^u) : K \times \mathbb{R}^u \times \mathbb{R}^k \rightarrow \mathbb{R}^u$ are continuously differentiable functions.

Further, we establish the links between a (weakly) efficient point of the initial multiobjective variational control model (P) and a (weakly) efficient point of the modified multiple objective variational control model $(P_{\kappa, \mu}(\bar{\varepsilon}, \bar{\varsigma}))$, constructed by considering the technique of modified cost functional. The main outcomes of the present study, that is, we state some characterization theorems for the considered multiple objective variational control models.

Theorem 31. *Let $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ be a feasible pair of the multiobjective variational control model (P) and the necessary efficiency conditions in Theorem 30 are satisfied at this point, with $\bar{\gamma} \in \mathbb{R}^q$ and $\bar{\xi}(\cdot) : K \rightarrow \mathbb{R}^m$ and $\bar{\nu}(\cdot) : K \rightarrow \mathbb{R}^u$. Also, we assume that $\int_{\alpha}^{\beta} \bar{\xi}^T g(n, \varepsilon, \varsigma) dn$ and $\int_{\alpha}^{\beta} \bar{\nu}^T [h(n, \varepsilon, \varsigma) - \dot{\varepsilon}] dn$ are univex at $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ with respect to Φ_g, b_g, κ, μ and Φ_h, b_h, κ, μ , respectively, where $s \leq 0 \implies \Phi_{g/h}(s) \leq 0$. If $\bar{\gamma} > 0$, then $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is an efficient point of the multiobjective variational control model with modified cost functional $(P_{\kappa, \mu}(\bar{\varepsilon}, \bar{\varsigma}))$ associated with the initial multiple objective variational control model (P) .*

Corollary 1. *Consider $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is an efficient point of the multiobjective variational control model (P) and the necessary efficiency conditions in Theorem 30 are satisfied at this point, with $\bar{\gamma} \in \mathbb{R}^q$ and $\bar{\xi}(\cdot) : K \rightarrow \mathbb{R}^m$ and $\bar{\nu}(\cdot) : K \rightarrow \mathbb{R}^u$. Further, assume that $\int_{\alpha}^{\beta} \bar{\xi}^T g(n, \varepsilon, \varsigma) dn$ and $\int_{\alpha}^{\beta} \bar{\nu}^T [h(n, \varepsilon, \varsigma) - \dot{\varepsilon}] dn$ are univex at $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ with respect to Φ_g, b_g, κ, μ and Φ_h, b_h, κ, μ , respectively, where $s \leq 0 \implies \Phi_{g/h}(s) \leq 0$. If $\bar{\gamma} > 0$, then $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is an efficient point of the*

multiobjective variational control model $(P_{\kappa,\mu}(\bar{\varepsilon}, \bar{\varsigma}))$ associated with the initial multiple objective variational control model (P) .

Next, to state that $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is a weakly efficient point to (P) , the hypothesis $\bar{\gamma} > 0$ can be removed.

Corollary 2. Consider $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is a weakly efficient point of the multiobjective variational control model (P) and the necessary efficiency conditions in Theorem 30 are satisfied at this point, with $\bar{\gamma} \in \mathbb{R}^q$ and $\bar{\xi}(\cdot) : K \rightarrow \mathbb{R}^m$ and $\bar{\nu}(\cdot) : K \rightarrow \mathbb{R}^u$. Further, assume that $\int_{\alpha}^{\beta} \bar{\xi}^T g(n, \varepsilon, \varsigma) dn$ and $\int_{\alpha}^{\beta} \bar{\nu}^T [h(n, \varepsilon, \varsigma) - \bar{\varepsilon}] dn$ are univex at $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ with respect to Φ_g, b_g, κ, μ and Φ_h, b_h, κ, μ , respectively, where $s \leq 0 \implies \Phi_{g/h}(s) \leq 0$. Then, the pair $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is a weakly efficient point of the multiobjective variational control model $(P_{\kappa,\mu}(\bar{\varepsilon}, \bar{\varsigma}))$ associated with the initial multiobjective variational control model (P) .

Remark 1. Let us notice that, without any univexity hypothesis imposed on the objective functional, we have formulated and proved the results in Theorem 31 and Corollaries 1 and 2.

In the following, we will state the converse results for the ones established above.

Theorem 32. Consider $(\bar{\varepsilon}, \bar{\varsigma})$ is an efficient point of the multiobjective variational control model $(P_{\kappa,\mu}(\bar{\varepsilon}, \bar{\varsigma}))$. Further, assume the functionals

$$\int_{\alpha}^{\beta} \rho^j(n, \varepsilon, \varsigma) dn, \quad j \in M,$$

are strictly univex at $(\bar{\varepsilon}, \bar{\varsigma})$ with respect to $\Phi_{\rho^j}, b_{\rho^j}, \kappa, \mu$, where $s \leq 0 \implies \Phi_{\rho^j}(s) \leq 0, j \in M$. Then, the pair $(\bar{\varepsilon}, \bar{\varsigma})$ is an efficient point of the original multiobjective variational model (P) .

Remark 2. Let us notice that, without any univexity assumptions imposed on the constraint functionals, we have stated the result formulated in Theorem 32.

If some weaker univexity hypotheses are imposed on the objective functional, then we can get the next results.

Theorem 33. Consider $(\bar{\varepsilon}, \bar{\varsigma})$ is a weakly efficient point of $(P_{\kappa,\mu}(\bar{\varepsilon}, \bar{\varsigma}))$. Further, assume the functionals $\int_{\alpha}^{\beta} \rho^j(n, \varepsilon, \varsigma) dn, j \in M$, are univex at $(\bar{\varepsilon}, \bar{\varsigma})$ on F with respect to $\Phi_{\rho^j}, b_{\rho^j}, \kappa, \mu$, where $s \leq 0 \implies \Phi_{\rho^j}(s) \leq 0, j \in M$. Then, the pair $(\bar{\varepsilon}, \bar{\varsigma})$ is a weakly efficient point to the original multiobjective variational model (P) .

Taking into account Corollary 1 and Theorem 32, we obtain the equivalence between the original multiple objective variational control model (P) and the modified multiple objective variational control model $(P_{\kappa,\mu}(\bar{\varepsilon}, \bar{\varsigma}))$, constructed by using the technique of modified cost functional.

Theorem 34. Consider the assumptions formulated in Theorems 31 and 32 are satisfied. Then, the pair $(\bar{\varepsilon}, \bar{\varsigma})$ is an efficient point of (P) if and only if it is an efficient point of the modified multiple objective variational control model $(P_{\kappa,\mu}(\bar{\varepsilon}, \bar{\varsigma}))$, constructed by using the technique of modified cost functional.

By Corollary 2 and Theorem 33 a similar result is valid.

Theorem 35. *Consider the assumptions formulated in Corollary 2 and Theorem 33 are satisfied. Then, the pair $(\bar{\varepsilon}, \bar{\varsigma}) \in F$ is a weakly efficient point of (P) if and only if it is a weakly efficient point of the modified multiple objective variational control model $(P_{\kappa, \mu}(\bar{\varepsilon}, \bar{\varsigma}))$, constructed by using the technique of modified cost functional.*

In addition, a numerical example is presented to illustrate the theoretical developments derived in the study.

The main novelty elements associated with this study are the following: (i) including the control variable in the context of univex variational problems with mixed constraints; (ii) introducing the concept of univexity for controlled simple integral functionals; (iii) characterizing efficient solutions for the considered class of problems by using the new concept of univexity and the modified objective functional technique. The main advantage of this work is, as mentioned above, the presence of the control function in the variational model considered. The inclusion of the control function in the context of univex variational problems with mixed constraints implies more laborious proofs because: the concept of (strictly) univexity associated with the simple integral type functional contains a new term; the problem under study includes the so-called *state equation*; the necessary efficiency conditions of Karush-Kuhn-Tucker include a new relation; the technique of modified cost functional for solving the non-convex multiobjective variational control model (P) contains a new term. All these facts make this research more widely applied in various branches of science.

The concept *multiobjective programming* is used to denote a type of optimization problem where two or more objectives are to be extremized (minimized or maximized) subject to certain constraints. The investigation of optimality conditions and/or duality for multiple objective variational control models has been one of the most attractive topics in the theory of nonlinear optimization. In consequence, the approach proposed in this chapter is a real benefit for the future research in the field of multiobjective variational optimization.

Concerning future research directions related to this study, we may point out the development of a framework for a non-smooth functionals using the subdifferential concept.

The content of this chapter is based on the study *On a class of multiple objective variational control models determined by univex functionals*, authored by Treanță, S., Ciontescu, M., Preeti, published in *Mathematical Control and Related Fields*, Vol. 18, 385-396, 2026; DOI: 10.3934/mcrf.2026001; WOS: 001627647600001; ISSN: 2156-8472 [121].

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